CHICAGO ARML

Chicago Area All-Star Math Team

2025 Tryout Solutions

Names in parentheses at the end of each problem indicate the problem's author. CAASMT thanks all the problem authors for their contributions to this year's tryout!

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1. Let $k = 9a + b$ where a and b are integers and $0 \leq b < 9$ (that is, a is the quotient and b the remainder on division of k by 9). Then $4k + 1 = 36a + 4b + 1$. Note that $36a$ is always divisible by 9, so $4k + 1$ is divisible by 9 if and only if $4b + 1$ is divisible by 9, and since $0 \leq b < 9$ this is only true when $b = 2$. That is, k is of the form $9a + 2$. This is a two-digit positive integer for $1 \leq a \leq 10$, so there are $\boxed{10}$ such numbers. (Holden Mui)

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2. First, note that $f(2) = 4$, $f_2(2) = f(4) = 0$, and every further application of f again obtains zero, so $f_{2024}(2) = 0$.

Next, $f(2 + \sqrt{3}) = 8 + 4\sqrt{3} - (7 + 4\sqrt{3}) = 1$. Then $f(1) = 3$, $f(3) = 3$, and the value never changes (that is, 3 is a fixed point of f), so $f_{2025}(2 + \sqrt{3}) = 3$.

Third, $f(2-\sqrt{3}) = 8-4\sqrt{3}-(7-4\sqrt{3}) = 1$. So, like the previous case, $f_{2026}(2-\sqrt{3}) = 3$.

Adding these results gives a total of $0 + 3 + 3 = \boxed{6}$. (Michael Caines)

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3. The number of diagonals of a convex n-gon is given by $\frac{1}{2}n(n-3)$. (This is because there are $\binom{n}{2} = \frac{1}{2}n(n-1)$ pairs of points that could be the ends of a diagonal, except that the n edges of the polygon itself must be subtracted from this.) So the required pairs (m, n) satisfy the equation

$$\frac{1}{2}m(m-3) - \frac{1}{2}n(n-3) = 19.$$

Multiplying both sides by 2 and rearranging terms yields $m^2 - n^2 + 3n - 3m = 38$. This equation can be factored as

$$(m-n)(m+n-3) = 38.$$

Since only integer solutions are desired, both $m - n$ and $m + n - 3$ must be factors of 38. There are only two ways to factor 38: $38 = 1 \cdot 38 = 2 \cdot 19$.

Setting $m - n = 1$ and $m + n - 3 = 38$ yields $m = 21$ and $n = 20$, while setting $m - n = 2$ and $m + n - 3 = 19$ requires $m = 12$ and $n = 10$. So the desired pairs are $\boxed{(21, 20) \text{ and } (12, 10)}$.

Alternate solution: in case you are not familiar with the formulas. Adding one new vertex to an n-gon adds $n-2$ new diagonals from the new vertex to each of the old vertices that are not adjacent to it, plus one diagonal between the two vertices between which the new vertex was placed—they used to be connected by a side, but they are no longer adjacent vertices so now that side becomes a diagonal. So adding one new vertex to an n-gon adds a total of $n-1$ new diagonals.

If $m = n+1$ then $n-1 = 19$ so $n = 20$ and $m = 21$, yielding the $(21, 20)$ solution. If $m = n+2$, then $(n-1)+n$ diagonals were added, so $2n-1 = 19$ and $n = 10$, so $m = 12$, yielding the $(12, 10)$ solution. It is simple to check that $(n-1)+n+(n+1) = 3n$, $(n-1)+n+(n+1)+(n+2) = 4n+2$, and so forth for adding more than two new vertices will never total 19 new diagonals, so those are the only solutions. (Michael Caines)

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4. First, factor 12! into primes. To determine the number of factors of a given prime are in $n!$, start with n and repeatedly divide by the prime, keeping the quotient and discarding the remainder, and add the quotients. Thus $12 \div 2 = 6$; $6 \div 2 = 3$; $3 \div 2 = 1$ (ignoring the remainder of 1). Adding, $6+3+1 = 10$ so there are 10 factors of 2 in 12!. Checking the other primes less than 12, the final factorization is

$$12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11.$$

Now, a number has exactly one trailing zero if it is divisible by 10, but not by 100. Thus, for a factor of 12! to have one trailing zero, it can have any number of 3's, 7's, and 11's, but it must have at least one 2, at least one 5, and *not* have two or more of both. That is, it can have from zero up to 5 (6 choices) factors of 3, 2 choices of number of factors of 7, and two choices each for factors of 11. If there is exactly one factor of 5, there are 10 choices of number of factors of 2 (zero factors of 2 is *not* a choice here!), but there must be exactly one factor of 2 if there are two factors of 5. That totals 11 possible choices in the numbers of factors of 2 and 5.

Applying the multiplication principle, the total number of divisors of 12! that have exactly one trailing zero is $11 \cdot 6 \cdot 2 \cdot 2 = \boxed{264}$. (Michael Caines, modified by Mark Fritz)

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5. A brute force solution uses the point-to-line distance formula. For the line with equation $Ax + By + C = 0$ and point (X, Y) the distance from the point to the line is given by $\dfrac{|AX + BY + C|}{\sqrt{A^2 + B^2}}$. The two lines have equations $12x - 5y = 0$ and $4x + 3y - 56 = 0$. The problem asks for points on the x-axis, so $y = 0$. Thus $|12x|/13 = |4x - 56|/5$. Carefully solving this absolute value equation gives two values for x: -91 and $13/2$. Adding these yields $\boxed{-169/2}$

A geometric approach is to note that the points equidistant to two lines are the points on the bisectors of the angles formed by the lines. These points form a pair of perpendicular lines which meet at the intersection of the two original lines. Let P be the point where the bisector of $\angle AXB$ meets the x-axis. Note distances $AX = 13$ and $BX = 15$. By the angle bisector theorem $AP/PB = AX/XB = 13/15$. Thus $AP = \frac{13}{13+15}AB = 13/2$, so P lies at $(13/2, 0)$. By the *external* angle bisector theorem, if P' is the point where the bisector of the *external* angle at X intersects meets the x-axis, then $P'A/P'B$ also equals 13/15. Thus, $P'A = \frac{13}{15}P'B = \frac{13}{15}(P'A + 28)$, so $\frac{2}{13}P'A = 28$ or $P'A = 91$. Therefore the x-coordinate of P' is -91. Adding the two x-coordinates results in the value above.

If you are unfamiliar with the external angle bisector theorem, an alternate approach is to note that line $\overleftrightarrow{XP}$ has slope -8 so the perpendicular line (the other lines of points equidistant to the original two lines) has slope 1/8. Since is passes through $(5, 12)$ its x-intercept is $(-91, 0)$. (Michael Caines)

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6. The metronome can clearly produce any number from 40 to 208 ticks per minute, if set to one tick per beat. If set to two ticks per beat, it can produce even numbers from 80 to 416, but the numbers from 80 to 208 are duplicates from one-tick settings. Similarly, it can produce multiples of 3 between 120 and 624, and multiple of 4 between 160 and 832. Count these systematically:

- From 40 to 208, there are $208 - 40 + 1 = 169$ setting.
- From 209 to 416 the setting must be a multiple of 2 or 3. There are 104 multiples of 2, and 69 multiples of 3 in this range, but 35 of these setting are multiples of both 2 and 3 so have been counted twice. By inclusion-exclusion, there are $104 + 69 - 35 = 138$ settings in this range.
- From 417 to 624 the setting must be a multiple of 3 or 4. There are 70 multiples of 3, 52 multiples of 4, and 18 multiples of both 3 and 4, resulting in $70 + 52 - 18 = 104$ more settings in this range.
- From 625 to 832 there are 52 multiples of 4.

The total is $169 + 138 + 104 + 52 = \boxed{463}$ settings. (Wilbert Chu)

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7. Let L be the graph of any relation $f(x, y) = 0$. Then to find the graph of $f(ax+b, y) = 0$, L must first be translated b units to the left, and then dilated by a factor of $1/a$. Similarly, $f(x, cy+d) = 0$ translates the original graph down by d units and then dilates the resulting graph by a factor of $1/c$.

Let $f(x, y) = 4x^3 + 2x - y^5 + 12$. According to the paragraph above,

$$\begin{aligned} g(x, y) &= f(3x - 4, -y - 1) \\ &= 4(3x - 4)^3 + 2(3x - 4) - (-y - 1)^5 + 12 \\ &= 4(27x^3 - 108x^2 + 144x - 64) + 6x - 8 + y^5 + 5y^4 + 10y^3 + 10y^2 + 5y + 1 + 12 \\ &= 108x^3 - 432x^2 + 576x - 256 + 6x - 8 + y^5 + 5y^4 + 10y^3 + 10y^2 + 5y + 1 + 12 \end{aligned}$$

This polynomial in x and y has integer coefficients with no common factors, and the sum of its coefficients is 38. So the desired quantity is $|38| = \boxed{38}$.

(A much simpler way to sum the coefficients of a polynomial is simply to evaluate the polynomial with all the variables set to 1. In this case, $g(1,1) = f(-1,-2) = -4-2+32+12 = 38$.)

Note: the absolute value was placed in this problem because the opposite polynomial also satisfies the requirements of the problem. (Michael Caines)

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8. Since Everett expects to earn 5/2 treats each day, he expects to answer 7/2 questions per day (5/2 correctly, and the last incorrectly). Now when the probability of success for each question is p, the expected number of trials before a failure is given by $p/(1-p)$, so the total number of expected trials is $1/(1-p)$. (Intuitively, if you expect to succeed 2/3 of the time and fail 1/3 of the time, on average there is one failure for every three trials, so you expect to fail every third try. The formula can be proven rigorously by summing the series for expected number of trials *before* a failure, $1-p(0+1p+2p^2+\cdots) = \dfrac{p}{1-p}$.

For Everett, $1/(1-p) = 7/2$, so $1-p = 2/7$ and $p = \boxed{5/7}$.

Alternate Solution: Before starting each day, Everett expects to earn $E = 5/2$ treats. He expects to answer the first question correctly with probability p, after which he would expect to earn E more treats. He also expects to answer the first question wrong with probability $1-p$, after which he would earn 0 treats. So $E = p(1+E) + (1-p)\cdot 0$. Thus $E(1-p) = p$. Setting $E = 5/2$ gives $\frac{5}{2}(1-p) = p$ or $\frac{5}{2} = \frac{7}{2}p$, so $p = 5/7$. (Michael Caines, modified by Micah Fogel)

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9. If Jack and Chris watch for three hours (which is four 45 minute intervals), the probability that they do *not* see a shooting star is $(1-1/2)^4 = 1/16$. Now if p is the probability that they see a shooting star in the next hour, the probability that they do *not* see a shooting star over a 3-hour period is $(1-p)^3$. So $(1-p)^3 = 1/16$. Therefore $1-p = (1/16)^{1/3}$, or $p = 1-(1/16)^{1/3} = \boxed{1-2^{-4/3}}$. (Holden Mui)

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10. To get a handle on this, start with a smaller seed, say $a_1 = 4$. The sequence would begin $4\ \ 1,1,2,1,3,1,4\ \ 2,2,3,2,4\ \ 3,3,4\ \ 4,5,1,\ldots$. Notice that a 1 occurs after the first occurrence of each other number and, in fact, can only occur after the first occurrence of a new number. Thus, there can't be more than four 1's until there is a 5. *The same is true for all other numbers up to 4.* Thus, in the first $4^2 = 16$ terms of the sequence, each number from 1 to 4 occurs exactly 4 times.

What happens after these 16 terms? The sequence continues

$$4,5,1,5,2,5,3,5,4,6,1,6,2,6,3,6,4,7,1,7,2,\ldots.$$

After 16 terms, in which each number occurred four times, and 4 was the last to occur, the next number is 4 (counting the four 4's) which is the fifth 4. A new number has occurred, which then sets off an increasing count of fives (the new number), until there are four of them, which is the *sixth* 4, which starts off an increasing count of sixes, and so on. Thus, there are 16 terms in which each number from 1 to 4 occurs four times, then a fifth 4, then blocks of $8 = 4 \cdot 2$ numbers counting each new number that comes along.

Similarly, if the sequence starts with 20, there are $20^2 = 400$ terms in which each number from 1 to 20 occurs twenty times, then a twenty-first 20, and then blocks of $20 \cdot 2 = 40$ numbers counting each new number. The first block counts 21's, then 22's, 23's, and 24's. The very next number will be the first 25. This occurs after $20^2 + 1 + 4 \cdot 40 = 561$ terms, so the first 25 will be term number $\boxed{562}$. (Wilbert Chu)

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11. A clever geometric argument shows that S is a circle of radius $3/4$. If interested, consult a reference on circle inversion. Here, a quick algebraic solution will show the same. Using the distance formula, $\sqrt{(x-1)^2 + y^2} = 3\sqrt{(x+1)^2 + y^2}$. Squaring both sides and collecting like terms yields $8x^2 + 20x + 8 + 8y^2 = 0$. Factoring out the 8 and completing the square yields $8((x + 5/4)^2 + y^2 - 9/16) = 0$ which is the equation of a circle centered at $(5/4, 0)$ and radius $3/4$. So the requested area is $\boxed{9\pi/16}$. (Michael Caines)

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12. First, compute the inner sum. Let t_1, t_2, and t_3 be the three complex roots of $x^3 = 3$. Then $(x - t_1)(x - t_2)(x - t_3) = x^3 - 3$. Substituting $-x$ for x and factoring out all the negative signs shows that $(x + t_1)(x + t_2)(x + t_3) = x^3 + 3$. Then

$$\frac{1}{s+t_1} + \frac{1}{s+t_2} + \frac{1}{s+t_3} = \frac{(s+t_2)(s+t_3) + (s+t_3)(s+t_1) + (s+t_2)(s+t_1)}{(s+t_1)(s+t_2)(s+t_3)}.$$

The numerator can be expanded to obtain $3s^2 + 2s(t_1 + t_2 + t_3) + (t_1t_2 + t_1t_3 + t_2t_3)$. Because the coefficients of x and x^2 in the polynomial $x^3 - 3$ are zero, by Viète's formulas the sum of t's and products of pairs of t's are both zero, so the numerator simplifies to $3s^2$. So the inner sum totals $\dfrac{3s^2}{s^3+3}$.

Now the two roots of $x^2 = 2$ are $s_1 = \sqrt{2}$ and $s_2 = -\sqrt{2}$. Substituting these and adding gives

$$\frac{3(\sqrt{2})^2}{(\sqrt{2})^3 + 3} + \frac{3(-\sqrt{2})^2}{(-\sqrt{2})^3 + 3} = \frac{6}{\sqrt{8} + 3} + \frac{6}{-\sqrt{8} - 3} = \frac{6 \cdot 6}{-8 + 9} = \boxed{36}.$$

(Holden Mui)

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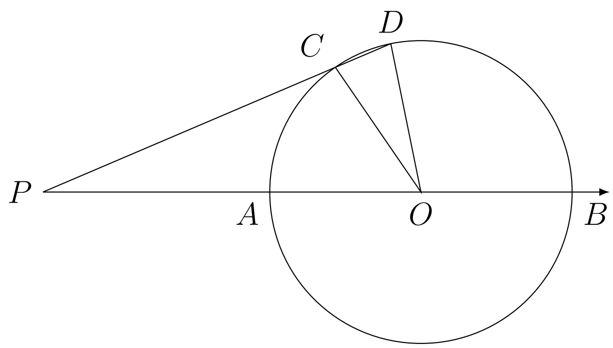
13. The configuration with this description is shown below:



Now $\triangle COD$ is isosceles, because two of its sides are radii of circle O. $\triangle DPO$ is similar to $\triangle COD$ because it is given that $\angle DPB \cong \angle DOC$ and both triangles share $\angle D$. So $\triangle DPO$ is also isosceles, and $PD = PO$. Finally, $PO = \frac{PA+PB}{2} = 5$, and the desired result is $PD = \boxed{5}$. (Michael Caines, modified by Micah Fogel)

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14. Set a coordinate system so that the car is initially at the origin, and it's initial direction is along the positive x-axis, and the unit is one kilometer. So before the car makes its first turn, it has reached the point $(2, 0)$. Now $60°$ to the right means the angle is $-60°$ so after three more kilometers, the car is now at $(2 + 3\cos(-60°), 0 + 3\sin(-60°)) = (3.5, -3\sqrt{3}/2)$. Another turn means the car is heading in a direction of $-120°$ and another 5 kilometers of travel puts it at $(3.5 + 5\cos(-120°), -3\sqrt{3}/2 + 5\sin(-120°)) = (1, -4\sqrt{3})$. This is $\sqrt{1 + 48} = \boxed{7}$ kilometers from the origin. (Holden Mui)

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15. Let the number be $\underline{a}\,\underline{b}\,\underline{c} = 100a + 10b + c$ where a, b, and c are the base-10 digits of the number. The sum of its digits is, of course, $a + b + c$. This can be rewritten as $(81+2\cdot 9+1)a+(9+1)b+c = 81a+9(2a+b)+(a+b+c)$, so the base-9 digits of the number are a, $2a + b$ and $a + b + c$. But then the sum of the digits would be $4a + 2b + c > a + b + c$ (unless a and b are zero, in which case the number does not have three digits in base 10). So there must be a carry from some position to the next. For instance if $a + b + c > 8$ the units digit in base 9 would be $a+b+c-9$ and the next digit would be $2a+b+1$. Unless $a+b+c-9$ was still larger than 8, in which case a second—and possibly a third—carry might have to take place. But note that each carry *decreases* one digit by 9 while increasing the next by 1, so the sum of digits in base 9 would be $4a+2b+c-8j$ where j is the total number of carries. Since this is supposed to equal the sum of digits in base 10, $4a + 2b + c - 8j = a + b + c$ or $3a + b = 8j$.

A similar analysis of the base-11 digits yields $100a+10b+c = (121-2\cdot 11+1)a+(11-1)b+c = 121a+11(-2a+b)+(a-b+c)$, whence the sum of the digits is c. Of course, this will be less than $a+b+c$ so at least one of these combinations must yield a negative "digit" and a borrow must occur. A borrow has the effect of increasing the digit by eleven, while decreasing the next digit by one. Overall, the sum of the digits increases by 10. That is, the sum of the digits of the number in its base-11 form will be $c+10k$. Thus, $a+b+c=c+10k$ or $a+b=10k$.

Now since a and b are digits in base 10, and a is not zero, the only possibility is that $a+b=10$. Then $2a+10=8j$, requiring a to be 3. Then $b=7$ and, to make the number as small as possible, $c=0$. The base-10 number is $\boxed{370}$, which equals 451_9 and 307_{11}, all of whose digital sums are 10. (Holden Mui)

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16. First, determine the areas of the pentagons which result from squares where the diagonal cuts off a triangle in the bottom right corner of the square. By symmetry, the total area of the pentagons will be twice this. Notice that the area of any particular pentagon is 1− the area of the triangle that is cut off. Also notice that all of these triangles are all similar to each other. The triangle in the bottom left square has base 1 and height 0.7, so its area is 0.35.

Now notice that the diagonal crosses the horizontal bottoms of the squares at $(0,0)$, $(10/7,1)$, $(20/7,2)$, $(30/7,3)$, and so on until it reaches $(70/7,7)=(10,7)$ in the upper right corner. The lengths of the bases of the triangles are the distance from where this intersection occurs to the next largest integer. Thus, the bases are $1, 4/7, 1/7, 5/7, 2/7, 6/7$, and $3/7$. Thus the areas of the triangles are $1\cdot 0.35$, $(4/7)^2\cdot 0.35$, $(1/7)^2\cdot 0.35$, and so forth. This totals

$$\frac{1^2+2^2+3^2+4^2+5^2+6^2}{7^2}\cdot 0.35=\frac{91}{49}\cdot 0.35=\frac{13}{20}.$$

So the total area of the pentagons is $6-0.65=5.35$. Doubling this to account for the pentagons that come from squares where the triangle is in the top left portion of the square yields $\boxed{10.7}$. (Is it just coincidence that the answer has a 10 and a 7 in it?) (Wilbert Chu)

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17. The units digit must be a 1, because when any nine-digit number starting with 2 or larger is mulitplied by 9, the result is a ten-digit number. And nine times any nine-digit number is at least 900,000,000, so the leading digit will be a 9. Thus, the number is of the form $9abcdefg1$ where each letter represents a single digit.

Continuing the analysis, g must be either 0 or 1. If it is 1, then the number is of the form $99bcdef11$. But $9\cdot 11fedcb99$ ends in 91, not 11. So $g=$. But for $9\cdot 10fedcba9$ to end in 01, $a=8$. So our number now looks like $98bcdef01$.

Now we want the number to be as large as possible, so for fun just try $bcdef=99999$. Lo and behold, $9\cdot 109{,}999{,}989=\boxed{989{,}999{,}901}$ and the number has been found! (Holden Mui)

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18. Since 6 across is a multiple of 50, the final digit in 4 down must be a zero. Since 4 down is a square, it must be 100, 400, or 900. The first digit of 1 down is thus 1, 4, or 9. The only cube starting with one of these digits is 125. So 1 down is 125 and 3 down is 100.

Now look at the second-to-last digit of 6 across. It must be either a 5 or a 0. But if it is a zero, then 3 down must also be 100, 400, or 900. But 5 across is a permutation of the digits of 2025, and the 0 has already been used in 4 down. So the second-to-last digit of 6 across is 5. Since there are no squares ending in 55, the middle digit of 3 down must be 2. This makes the second digit of 5 across the remaining unused digit of 2025, namely the 5.

3 down is a square ending in 25. Thus it is 225 or 625. That makes the first two digits in 2 down either 25 or 65. Since there are no perfect squares starting with 65 ($25^2 = 625$, $26^2 = 676$), the final pieces fall into place. 2 down must be 225 and the solution to the puzzle is

1	2	2	1
2	5	2	0
5	6	5	0

. (Chris Jeuell)

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19. For a polygon with n vertices, there are $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$ sets of three distinct vertices. A regular polygon can be inscribed in a circle. Now a right triangle inscribed in a circle has a diameter of the circle for its hypotenuse. Thus, if the three chosen vertices are the vertices of a right triangle, two of them are diametrically opposite each other, while the third could be any remaining vertex of the polygon. This can only happen if n is even, in which case there are $\frac{n}{2}$ opposite pairs of vertices, with each pair having $n-2$ remaining vertices to choose from to form the right angle. Thus, when n is even there are $\frac{1}{2}n(n-2)$ sets of vertices that form a right triangle. So the probability that three distinct vertices chosen from a regular n-gon form a right triangle is

$$\frac{\frac{1}{2}n(n-2)}{\frac{1}{6}n(n-1)(n-2)} = \frac{3}{n-1}.$$

This is between $\frac{1}{5}$ and $\frac{1}{3}$ inclusive when n is between 10 and 16, inclusive. Recalling that n must also be even, the total of such values of n is $10+12+14+16 = \boxed{52}$. (Michael Caines)

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20. If you are familiar with modular arithmetic, you would quickly note that that 1–10 represent each non-zero remainder modulo 11, and each is relatively prime to 11, so their reciprocals also represent each non-zero remainder. Adding these reciprocals then gives 55, which is divisible by 11, so the answer is zero.

If you are not familiar with modular arithmetic, pair up the fractions as follows:

$$\frac{1}{1}+\frac{1}{10}=\frac{11}{10};\frac{1}{2}+\frac{1}{9}=\frac{11}{18};\frac{1}{3}+\frac{1}{8}=\frac{11}{24};\frac{1}{4}+\frac{1}{7}=\frac{11}{28};\frac{1}{5}+\frac{1}{6}=\frac{11}{30}.$$

Now all the numerators are 11, and adding these pairs together will result in a fraction whose numerator is divisible by 11. Since none of the denominators are divisible by 11, when written in lowest terms the resulting fraction's numerator is still divisible by 11, so the remainder on division by 11 must be $\boxed{0}$. (Holden Mui)

21. Call the three creature types contained in the first pack of cards Fritz purchased the *core group*. Since three creature typs are represented in the core group, the problem becomes to determine the probability that the next two packs contain all three of the types that are not in the core grouyp. Note that there are $\binom{6}{3} = 20$ choices of three creature types for any pack.

Now in the second pack that Fritz purchased, the three creature types in that pack could overlap with the creature types in the core group in four different ways:

- One possibility is that there is no overlap at all, and all three creature types in the second pack are different from those in the core group. This can happen in only one way, as there are exactly three types that are *not* in the core group, and the second pack must have precisely these three types in it. In this case, after two packs, each of the six creature types has been represented, and the combination of creatures in the third pack is irrelevant—all $\binom{6}{3} = 20$ possible combinations of creatures in the third pack give at least one of each type represented in the first three packs. This gives a total of $1 \cdot 20 = 20$ combinations of creature types in the first three packs that allow all six to be represented in those packs.

- The next possibility for the second pack is that two creature types are different than the core group, but one of the creatures in the second pack was already in the core group. This can happen in nine different ways—there are three choices of which creature overlaps the core group, and three choices of which non-core creature is not included among the first two packs. So in the first two rounds, five types are represented. Of the third round choices, there are $\binom{5}{3} = 10$ ways for all three creatures in that pack to be among those 5, and there are $\binom{5}{2} \cdot \binom{1}{1} = 10$ ways in which two of the three could be among the 5, with the third type of creature in the third pack being the sixth and final creature type that had not been in the first two packs. Thus, there are $9 \cdot 10 = 90$ combinations of creatures in these three packs that allow all six types to be represented after three rounds, when only five of the types have representatives after two rounds.

- Next, the second round could end up giving only one new creature type. There are nine such choices (three choices of core type that is not duplicated, three choices of new type that is represented). For each of these, there are only four choices for the third pack to fill in all the missing types (both unseen types must be in the pack, and the third type in the pack can be any of the four types that have shown up in the first two packs. The total here is $9 \cdot 4 = 36$ combinations of creature types allows the three packs to represent all six types of creatures.

- Lastly, if only three creatures are represented after the first two packs—the second pack contained exactly the same creatures as the first pack—the third pack must contain all three types not in the first two packs. Clearly there is only one way for the second round to exactly duplicate the first round, and one possibility that the third pack contains all three remaining types. This leads to just one more combination.

So the total number of combinations of choices where all six types are represented after three rounds is $20+90+36+1 = 147$. On the other hand, once the core group is determined, there are $20 \cdot 20 = 400$ ways to make the second- and third-round choices of creatures. So the final probability is $\boxed{147/400}$. (Mark Fritz)

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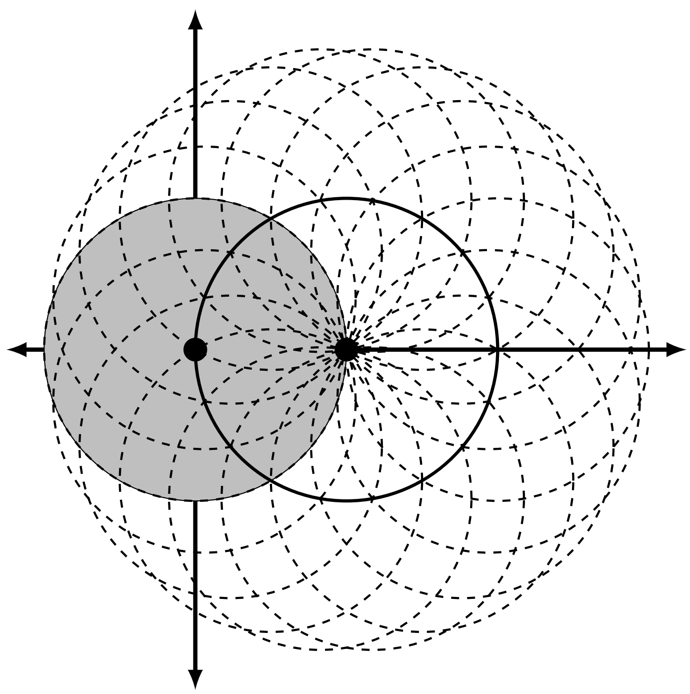
22. Set up a coordinate system so that Froggy starts at the origin, and is at $(1, 0)$ after the first jump. After the second hop, Froggy is equally likely to be at any point on the circle of radius 1 centered at $(1, 0)$, shown as the thick solid circle in the diagram below. After the second hop, Froggy is equally likely to be at any point that is on a circle of radius 1 centered at any point of the previous circle. These are the points on the dashed circles below. The shaded circle consists of the points within 1 meter of Froggy's starting position (a unit disk centered at the origin).



The dashed circles fill up a larger circle of radius 2 centered at $(1, 0)$, which has area 4π. The points within one meter of Froggy's starting position is a unit disk (area π) entirely inside the larger circle. Since $\frac{\pi}{4\pi} = \frac{1}{4}$, and Froggy is equally likely to be anywhere within the larger circle, the probability that Froggy is within one meter of the starting point is $\boxed{1/4}$.

An alternate approach is to measure the actual angles that indicate the direction of Froggy's hops. Begin by setting up the same coordinate system as before. To make the second hop, Froggy chooses a random number α between 0 and 2π and hops one meter in the direction that makes an angle of α with the positive x-axis. The third hop is then accomplished by choosing another direction, β and hopping one meter in that direction. At the end of these two hops, Froggy's position is $(1 + \cos(\alpha) + \cos(\beta), \sin(\alpha) + \sin(\beta))$.

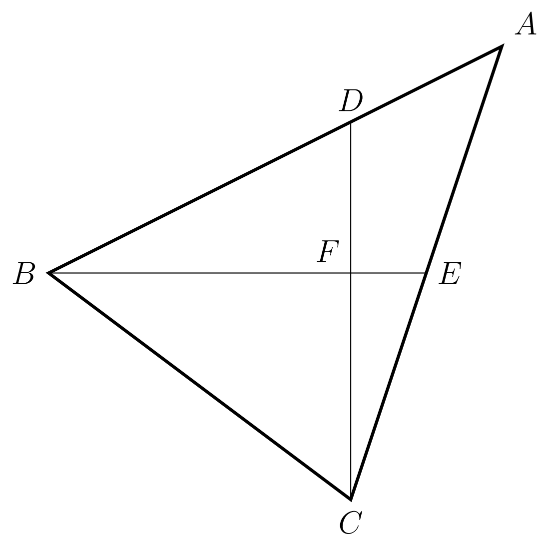
Using the distance formula, the square of Froggy's distance from the origin is

$$\begin{gathered}1 + 2\cos(\alpha) + 2\cos(\beta) + \cos^2(\alpha) + \cos^2(\beta) + 2\cos(\alpha)\cos(\beta) \\ + \sin^2(\alpha) + 2\sin(\alpha)\sin(\beta) + \sin^2(\beta) \\ = 3 + 2(\cos(\alpha) + \cos(\beta) + \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ = 3 + 2(\cos(\alpha) + \cos(\beta) + \cos(\alpha - \beta))\end{gathered}$$

Thus, the distance will be at most one when $s = \cos(\alpha) + \cos(\beta) + \cos(\alpha - \beta) \leq -1$. By symmetry, let $\alpha \geq \beta$. When $\alpha = \pi$, Froggy has jumped back to the origin with the second hop, and now any choice of β leaves Froggy within one meter of the starting point. Since $\alpha \geq \beta$ the line segment from $(\pi, 0)$ to (π, π) in the $\alpha\beta$-plane consists entirely of points where $s = -1$. From the point $\beta = \pi$, α is free to be any value, so α may now increase to be up to 2π. From the point $(2\pi, \pi)$ in the plane, any point along the segment from there back to $(\pi, 0)$ (that is, along the line $\beta = \alpha - \pi$) also maintains constant value $s = -1$. So within the triangle $0 \leq \alpha \leq 2\pi$ and $0 \leq \beta \leq \alpha$ which has area $\frac{1}{2} \cdot 2\pi \cdot 2\pi = 2\pi^2$ the area of points where $s \leq -1$ is within the triangle with vertices $(\pi.0)$, (π, π), and $(2\pi, \pi)$ which has area $\frac{1}{2} \cdot \pi \cdot \pi = \frac{\pi^2}{2}$. Taking the ratio of these areas once again gives an answer of 1/4. (Holden Mui)

23.

A diagram of the figure appears below:



Now choose units to that $CF = 3$ and $FB = 4$. Since $\triangle CFB$ is a right triangle, $BC = 5$. Perhaps the simplest approach is to use mass points. Place weight 10 at A, 5 at B and 10 at C. The triangle is now balanced along the cevians $\overline{BE}$ and $\overline{CD}$. Since the triangle is balanced along $\overline{BE}$ the two weights on side $\overline{AC}$ can be moved to point E and the triangle will remain balanced. Now B has a weight of 5 and E has weight 20, so to be balanced $BF : FE = 4 : 1$, so $FE = 1$. Then, in right triangle CFE, hypotenuse $EC = \sqrt{10}$. Since E is the midpoint of $\overline{AC}$, AE also has length $\sqrt{10}$, so $AC = 2\sqrt{10}$.

Returning the weights to the original vertices and then moving the weights on side $\overline{AB}$ to point D determines that $DF : FC = 2 : 3$, so $FD = 2$. From right triangle DFB, the Pythagorean Theorem now renders $DC = 2\sqrt{5}$ so $AB = 3\sqrt{5}$.

Finally, the law of cosines can be used:

$$\begin{aligned} BC^2 &= AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos(A) \\ 25 &= 45 + 40 - 2\sqrt{40 \cdot 45} \cdot \cos(A) \\ -60 &= -60\sqrt{2} \cdot \cos(A) \end{aligned}$$

So $\cos(A) = \boxed{\sqrt{2}/2}$. Yes, A is a 45° angle.

While a well-prepared mathlete should know the method of mass points, if you haven't learned it yet an alternate solution could run as follows.

To ensure that they are prependicular, cevian $\overline{BE}$ will lie along the x-axis and cevian $\overline{CD}$ along the y-axis. Their intersection, F, will be located at the origin. Then to get the ratio $CF : FB = 3 : 4$, locate B at $(-4, 0)$ and C at $(0, -3)$. Now to ensure that E is the midpoint of $\overline{AC}$ the y-coordinate of A must be 3, while to cause $\overline{AB}$ to be divided in a $1 : 2$ ration the x-coordinate of A must be 2. So A is located at $(2, 3)$. The law of cosines can now be applied to $\triangle ABC$ as before, or note that segments $\overline{BA}$ and $\overline{CA}$ have slopes $1/2$ and 3 respectfully. Then the formula for the tangent of a difference yields $\tan(A) = 1$, making A a 45° angle and thus $\cos(A) = \sqrt{2}/2$. (Mark Fritz)

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24. Note that the domain of this function is $[8, \infty)$. For any x in this domain, the laws of logarithms allow the logarithms to be combined:

$$f(x) = \log_b((x + \sqrt{x^2 - 64})(x - \sqrt{x^2 - 64})).$$

Multiplying the terms inside the logarithm yields $x^2 - (\sqrt{x^2 - 64})^2 = x^2 - (x^2 - 64) = 64$. That is, $f(x)$ is *constant* on its domain. So the question simplifies to finding the number of positive rational numbers b for which $\log_b(64)$ is an integer. Let this integer be k. Then $b^k = 64$. The following combinations work: $2^6 = 4^3 = 8^2 = 64^1$ and $(\frac{1}{2})^{-6} = (\frac{1}{4})^{-3} = (\frac{1}{8})^{-2} = (\frac{1}{64})^{-1} = 64$. There are $\boxed{8}$ possible choices for b. (Micah Fogel)